

Equations of one-dimensional hydrodynamics with quantum thermal fluctuations taken into account

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Abstract

We propose a generalization of equations of quantum mechanics in the hydrodynamic form by introducing the terms taking into account the diffusion velocity at zero and finite temperatures and the density energy of diffusion pressure of the thermal vacuum into the Lagrangian density. Based on this, for a model of one-dimensional hydrodynamics, we construct a system of equations that are similar to the Euler equations but taking quantum and thermal effects into account. They are a generalization of equations of Nelson's stochastic mechanics and can be used to describe a new matter state, namely, nearly perfect fluidity.

Key words : (\hbar, k) dynamics, quantum thermostat, cold vacuum, thermal vacuum, effective action, self-diffusion, drift and diffusion velocities

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1. Introduction

The explanation of the universal property of matter, namely, nearly perfect fluidity (NPF) [1], [2], discovered experimentally in different media, such as quark–gluon plasma, ultracold gases in traps, liquid helium, and even graphene, have attracted the attention of many researchers in the last several years [3]. Particular theoretical approaches were used for this purpose. However, there is no explicit basis combining them in the published papers. Only

one fact makes them close to each other, namely, the presence of the combination of the Planck and Boltzmann world constants in the form $\varkappa = \hbar/2k_B$ in the final result compared with the experimental data.

Irrespective of the NPF study, it was established that this quantity appears in the description of many quantum thermal phenomena, including transport processes. In addition, the characteristic of the effective environmental ("thermal" vacuum) influence in the equilibrium state is expressed in terms of \varkappa in (\hbar, k) dynamics, which is a modification of quantum theory at finite temperatures [4], [5].

Because the case in point is the description of the entire class of phenomena in which the nonadditivity of influences of the quantum and thermal types is manifested, the theory to be developed must be independent of particular concepts of medium structure as much as possible, which is typical of thermodynamics. In particular, to solve the NPF problem, hydrodynamics as a part of nonequilibrium thermodynamics must be generalized, primarily, to the quantum region. Although fluctuations in hydrodynamics have been taken into account during the last fifty years [6], at present, there is no consistent theory that takes quantum and thermal effects into account simultaneously.

In this paper, we propose to construct the holistic stochastic hydrodynamics in a new way by starting from the microdescription. To do this, we proceed from the hydrodynamic form of quantum mechanics, which is well known at zero temperature. We generalize it naturally to the case in which self-diffusion in the cold and thermal vacua are taken into account explicitly. For the first time, this allows extending the hydrodynamic form of quantum mechanics to finite temperatures and taking into account not only self-diffusion manifested at zero temperature too, but also the energy density of the diffusion pressure of the thermal vacuum.

As a result, for a one-dimensional model, we obtained a system of equations of stochastic hydrodynamics that is applicable at any temperature. Its distinction from the standard theory is that it takes quantum and thermal fluctuations into account nonadditively. Moreover, we managed to write these equations in the form of equations of two-velocity hydrodynamics, which is a generalization of Nelson's stochastic mechanics.

2. Effective action as a universal characteristic of transport processes

We recall that in the framework of (\hbar, k) dynamics, we introduced a new macroparameter, namely, the effective action of the quantum thermostat on a system [5]

$$\mathbb{J} = \frac{\hbar}{2} \sqrt{\alpha^2 + 1} = \mathbb{J}^0 \Upsilon, \quad (1)$$

where

$$\alpha^2 \equiv \sinh^{-2} \varkappa \frac{\omega}{T}; \quad \Upsilon = \coth(\varkappa \frac{\omega}{T}),$$

and $\mathbb{J}^0 = \hbar/2$ is the limiting values of \mathbb{J} at the Kelvin temperature $T \rightarrow 0$.

The most important macroparameters (the effective temperature, the effective internal energy, and the effective entropy) are expressed in terms of this quantity in the equilibrium case:

$$\mathbb{T} = \frac{\omega}{k_B} \mathbb{J}; \quad (2)$$

$$\mathbb{U} = \omega \mathbb{J}; \quad (3)$$

$$\mathbb{S} = \mathbb{S}^0 \left\{ 1 + \ln \frac{\mathbb{J}}{\mathbb{J}^0} \right\}, \quad (4)$$

where $\mathbb{S}^0 = k_B$ is the limiting value of \mathbb{S} as $T \rightarrow 0$.

In this connection, it would be natural to assume the limiting value of the ratio of two fundamental quantities, namely, the effective action \mathbb{J} and the effective entropy \mathbb{S} ,

$$\varkappa = \frac{\hbar}{2k_B} \equiv \lim_{T \rightarrow 0} \frac{\mathbb{J}}{\mathbb{S}} \quad (5)$$

to be the physical definition of the universal constant \varkappa .

However, this is not all. As shown below, effective transport coefficients, which are typical of nonequilibrium thermodynamics, can also be expressed in terms of this quantity. The latter can be seen if the self-diffusion process occurring in a medium with the inhomogeneous density after the temperature equilibrium is established is used as an example.

Indeed, it was shown in the theory of Brownian motion at sufficiently high temperatures [7] that the uncertainties relation of the form

$$(\Delta p) (\Delta q) = m D_T \quad (6)$$

is valid in this case (for $t \gg \tau$). Here, D_T is the coefficient of purely thermal diffusion; in this case, $D_T = k_B T \tau / m$ for a free microparticle (τ is the relaxation time) and $D_T = k_B T / m \omega$ for a Brownian oscillator [8].

As shown in [9], the Schrödinger uncertainties relation for the quantum oscillator in the state of equilibrium with the thermal vacuum has the form

$$(\Delta p) (\Delta q) = \mathbb{J} = \left(\frac{\hbar}{2}\Upsilon\right). \quad (7)$$

Comparing (6) and (7), we write this relation in the form

$$(\Delta p) (\Delta q) = m D_{ef}. \quad (8)$$

Then it is natural to call the quantity

$$D_{ef} = D_{qu} \Upsilon \equiv \frac{\mathbb{J}}{m} \quad (9)$$

the effective self-diffusion coefficient or the effective action on the mass unit and the quantity $D_{qu} = \hbar/2m$ the quantum diffusion coefficient in the cold vacuum. The quantity D_{qu} was introduced previously, in particular, by Nelson [10].

Obviously, the coefficient D_{ef} acquires the physical meaning of the effective action per mass unit. The limiting values of D_{ef} at high ($k_B T \gg \hbar \omega / 2$) and low ($k_B T \ll \hbar \omega / 2$) temperatures are equal to

$$D_{ef} \rightarrow D_T = k_B T / m \omega; \quad D_{ef} \rightarrow D_{qu} \equiv \frac{\hbar}{2m}, \quad (10)$$

respectively.

Proceeding from relation (8), all other effective transport coefficients can be expressed in terms of \mathbb{J} . We recall that according to the kinetic theory, the diffusion coefficient D_T has the meaning of the kinematic viscosity coefficient so that for shear viscosity, the formula

$$\eta = D_T \rho_m,$$

where ρ_m is the bulk mass density, is valid.

Assuming that the interrelation between the shear viscosity coefficient η_{ef} and the effective self-diffusion coefficient D_{ef} of form (9) has a similar form, we obtain

$$\eta_{ef} = D_{ef} \rho_m = \frac{\mathbb{J}}{V}, \quad \frac{1}{V} \mathbb{C}_V D_{ef}. \quad (11)$$

Thus, η_{ef} has the physical meaning of the specific effective action.

In our opinion, the ratio of the heat conductivity coefficient to the conduction coefficient in the Wideman–Franz law

$$\frac{\lambda}{\sigma} = \gamma \left(\frac{k_B}{e} \right)^2 T = \gamma \frac{k_B}{e^2} (k_B T), \quad (12)$$

where γ is a numerical coefficient, is also interesting. It is obvious that the presence of the factor $k_B T$ in this formula means that the classical thermostat model was initially assumed in this law.

In our opinion, the generalization of this law to the quantum thermostat model must have the form

$$\frac{\lambda_{ef}}{\sigma_{ef}} = \gamma \frac{k_B}{e^2} (k_B \mathbb{T}) = \gamma \frac{k_B}{e^2} \omega \mathbb{J} = \gamma \frac{k_B}{e^2} \omega m D_{ef}. \quad (13)$$

Thus, the majority of transport coefficients can be expressed in terms of the effective self-diffusion coefficient D_{ef} , which can, in principle, be obtained experimentally.

As for the constant \varkappa , in the analysis of particular experiments, it can be expressed in terms of observed transport coefficients by relations of the type

$$\varkappa = \left(\frac{D_{ef}}{\mathbb{S}/m} \right)_{min} = \left(\frac{\eta_{ef}}{\mathbb{S}/V} \right)_{min} = \dots, \quad (14)$$

where \mathbb{S}/m is the effective entropy of the mass unit and \mathbb{S}/V is the effective entropy of the volume unit.

3. Standard quantum mechanics in the hydrodynamic form

The standard quantum mechanics in the nonrelativistic field form (at $T = 0$) can be obtained if the variation of the action functional

$$\mathcal{S} = \int_{t_1}^{t_2} dt \int dq \mathcal{L}_0[\psi^*; \psi] \quad (15)$$

becomes zero [11]. Here, $\mathcal{L}_0[\psi^*; \psi]$ is the Lagrangian density for one spinless particle at $T = 0$, and $\psi(q, t)$ and $\psi^*(q, t)$ are the wave function and its complex conjugate function, which have the meaning of independent nonrelativistic fields. (We restrict ourselves to the one-dimensional case.)

It is obvious that the functional $\mathcal{L}_0[\psi^*; \psi]$ in the general case must be chosen in the form

$$\mathcal{L}_0[\psi^*; \psi] = \psi^*(q, t) \left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} \right) \psi(q, t) - \psi^*(q, t) U(q) \psi(q, t), \quad (16)$$

where the nonrelativistic limit of the Klein–Gordon operator is in the parentheses on the right, and the potential energy operator $U(q)$ characterizes the regular influence energy.

Independently varying the action of form (15) with respect to the field ψ^* leads to the condition

$$\int_{t_1}^{t_2} dt \int dq \frac{\delta \mathcal{L}_0[\psi^*; \psi]}{\delta \psi} = \int_{t_1}^{t_2} dt \int dq \left(i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} - U(q) \psi \right) = 0, \quad (17)$$

which leads to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} + U(q) \psi. \quad (18)$$

Accordingly, its complex conjugate equation is obtained by varying the action of form (15) with respect to ψ and differs from formula (18) by the replacement of i with $-i$ and ψ with ψ^* . We stress that the Schrödinger equations for the complex wave functions ψ and ψ^* have the meaning of the Lagrange–Euler equations; in this case, the wave functions are always complex in the full-scale quantum mechanics.

We now represent the wave function in the form

$$\psi(q, t) = \sqrt{\rho(q, t)} \exp\{i\theta(q, t)\}, \quad (19)$$

where $\rho(q, t) = |\psi(q, t)|^2$. This expression, together with its complex conjugate expression, can be substituted directly in the Schrödinger equations for ψ and ψ^* , and the system of equations for the functions $\rho(q, t)$ and $\theta(q, t)$,

which is well known in published papers as quantum mechanics in the hydrodynamic form, can be obtained [11],[12].

Because our aim is to construct modified hydrodynamics based on the microdescription, we propose another approach to the problem. It requires developing the theory in the Lagrange formulation from the very beginning. Therefore, we start from the transformation of the Lagrangian density \mathcal{L}_0 by passing to variables that are more adapted to the hydrodynamic description. As functional arguments of the Lagrangian density, we chose two independent real functions, namely, the probability density ρ and the phase θ instead of the complex wave functions ψ and ψ^* . They are similar to the functions of the mass density ρ_m and the drift velocity $v \sim \frac{\partial \theta}{\partial q}$, which are typical of the standard hydrodynamics.

To do this, we replace the arguments in Lagrangian density (16) by substituting expression (19) and the corresponding expression for ψ^* in it. After the substitution, we obtain

$$\begin{aligned} \mathcal{L}_0[\psi; \psi^*] = \mathcal{L}_0[\rho; \theta] = & -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{\hbar^2}{2m} \left(\frac{\partial \theta}{\partial q} \right)^2 \rho - \frac{\hbar^2}{8m} \left(\frac{\partial \rho}{\partial q} \right)^2 \frac{1}{\rho} - \\ & -U(q)\rho + i \frac{\hbar}{2} \frac{\partial \rho}{\partial t} + \frac{\hbar^2}{2m} \cdot \frac{\partial}{\partial q} \left(\frac{1}{2} \frac{\partial \rho}{\partial q} + i \rho \frac{\partial \theta}{\partial q} \right) \end{aligned} \quad (20)$$

Here, the term containing $\frac{\partial \rho}{\partial t}$ can be neglected because its contribution is zero after variation of the action \mathcal{S} of form(15) with respect to θ and ρ . The last term in (20) is the total derivative with respect to q , so that it can also be excluded from the definition of $\mathcal{L}_0[\rho; \theta]$. Therefore, as the expression for the Lagrangian density $\mathcal{L}_0[\rho; \theta]$, we finally assume

$$\mathcal{L}_0[\rho; \theta] = -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{\hbar^2}{2m} \left(\frac{\partial \theta}{\partial q} \right)^2 \rho - \frac{\hbar^2}{8m} \left(\frac{\partial \rho}{\partial q} \right)^2 \frac{1}{\rho} - U(q)\rho. \quad (21)$$

Varying the action \mathcal{S} of form (15), in which $\mathcal{L}_0[\rho; \theta]$ now has form (21), with respect to the variables θ and ρ successively, we obtain the equations for the real functions $\rho(q, t)$ and $\theta(q, t)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} \left(\rho \frac{\hbar}{m} \frac{\partial \theta}{\partial q} \right) = 0, \quad (22)$$

$$\hbar \frac{\partial \theta}{\partial t} + \frac{\hbar^2}{2m} \left(\frac{\partial \theta}{\partial q} \right)^2 + U(q) - \frac{\hbar^2}{8m} \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial q} \right)^2 + 2 \frac{\partial}{\partial q} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q} \right) \right] = 0. \quad (23)$$

These equations coincide with the equations that could be obtained for the functions ρ and θ directly from the Schrödinger equations. However, it is now clear that they have the meaning of the Lagrange–Euler equations for the action \mathcal{S} of form (15) expressed in terms of the variables ρ θ .

It is assumed traditionally that Eq. (22) is the continuity equation for $\rho(q, t)$. In its turn, Eq. (23) taking into account that the quantity $\hbar\theta(q, t)$ has the dimension of action is an analogue of the Hamilton–Jacobi equation. In this case, the term in brackets in formula (23) is interpreted sometimes as an additional energy $U_{qu}(q)$ of quantum nature vanishing in the semiclassical limit as $\hbar \rightarrow 0$.

Of course, Eqs. (22) and (23) for ρ and θ and the Schrödinger equations for ψ and ψ^* are equivalent formally. However, the derivation of equations of quantum mechanics in hydrodynamic form (22) and (23) directly from the principle of least action is physically more preferable in the construction of stochastic hydrodynamics. At the same time, to obtain the sought result, the problem of the form of the Lagrangian density must be solved. In our opinion, it must take the stochastic environmental influence (quantum thermostat) into account consistently.

4. Quantum self-diffusion in the "cold" vacuum

To reveal the possibilities of generalizing $\mathcal{L}_0[\rho; \theta]$, we first consider the case of the cold vacuum. To do this, we make the second and third terms in expression (21) have the physical meaning. According to the terminology introduced by Kolmogorov [13] for the Markov processes in the general theory of stochastic processes and used by Nelson [10] in his stochastic mechanics, we call the quantity

$$v \equiv \frac{\hbar}{m} \frac{\partial \theta}{\partial q} \quad (24)$$

the drift velocity. Accordingly, we call the quantity

$$u \equiv -D_{qu} \frac{1}{\rho} \frac{\partial \rho}{\partial q} = -\frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial \rho}{\partial q} \quad (25)$$

the diffusion velocity in the cold vacuum.

If the velocities v and u are used, formulas (21)–(23) can be written in the forms

$$\mathcal{L}_0[\rho, \theta] = -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{m}{2} (v^2 + u^2) \rho - U \rho; \quad (21a)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} (\rho v) = 0; \quad (22a)$$

$$\hbar \frac{\partial \theta}{\partial t} + \frac{m}{2} v^2 + U - \frac{m}{2} \left[u^2 - \frac{\hbar}{m} \frac{\partial u}{\partial q} \right] = 0, \quad (23a)$$

which open opportunities for generalization.

It follows from formula (22) that standard continuity equation (22) is quasiclassical because the probability flux density in it depends only on the drift velocity v , while the diffusion velocity u produced by the stochastic action of the cold vacuum is not taken into account in it.

In this connection, we recall that the Fokker–Planck equation [14]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} (\rho V) = 0, \quad (26)$$

which contains the total velocity of the probability flux density

$$V = v + u, \quad (27)$$

is a more general continuity equation according to Kolmogorov [13]. We show that it allows describing the approach to the thermal equilibrium state because of self-diffusion in the cold vacuum too.

We call attention to the fact that the combination $\frac{m}{2}(v^2 + u^2)$ in expression (21) for $\mathcal{L}_0[\rho, \theta]$ is the sum of independent contributions of the kinetic energies of the drift and diffusion motions. At the same time, the probability flux depends on the total velocity of form (27). In connection with this, to obtain the Fokker–Planck equation in expression (21), the natural replacement of $(v^2 + u^2)$ with V^2 suggests itself, which allows taking into account the total expression for the kinetic energy related to the probability flux. Thus, even the standard quantum mechanics (at $T = 0$) admits the opportunities for generalization.

Thus, we generalize the Lagrangian density $\mathcal{L}_0[\rho, \theta]$ of form (21) using the corresponding replacement. Then we obtain

$$\begin{aligned} \tilde{\mathcal{L}}_0[\rho, \theta] &= -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{m}{2} V^2 \rho - U \rho = \\ &= \mathcal{L}_0[\rho; \theta] - m v u \rho = \mathcal{L}_0[\rho; \theta] + \frac{\hbar^2}{2m} \frac{\partial \theta}{\partial q} \frac{\partial \rho}{\partial q}. \end{aligned} \quad (28)$$

Varying the action functional \mathcal{S} of form (15) with $\tilde{\mathcal{L}}_0[\rho, \theta]$ with respect to θ automatically leads to the Fokker–Planck equation with the quantum diffusion coefficient

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q}(\rho V) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} \left(\rho \frac{\hbar}{m} \frac{\partial \theta}{\partial q} \right) - D_{qu} \frac{\partial^2 \rho}{\partial q^2} = 0. \quad (29)$$

At the same time, varying \mathcal{S} with respect to ρ remains the Hamilton–Jacobi equation almost unchanged; one insignificant term

$$\hbar \frac{\partial \theta}{\partial t} + \frac{m}{2} v^2 + U - \frac{m}{2} \left(u^2 - \frac{\hbar}{m} \frac{\partial u}{\partial q} \right) + \frac{\hbar}{2} \frac{\partial v}{\partial q} = 0 \quad (30)$$

appears in the Hamilton–Jacobi equation compared with (23a).

Obtained system of equations (29) and (30) generalizes Eqs. (22) and (23), allowing one to take the quantum stochastic influence of the cold vacuum into account consistently.

5. Self-diffusion in the quantum thermostat for $T \neq 0$

We now use our developed approach to the description of self-diffusion with the simultaneous inclusion of the quantum and thermal effects. For this purpose, we introduce the Lagrangian density $\tilde{\mathcal{L}}_T[\rho, \theta]$, which is temperature-dependent, and demand that it transform into the expression $\tilde{\mathcal{L}}_0[\rho, \theta]$ of form (28) as $T \rightarrow 0$. To do this, it suffices to replace the diffusion coefficient D_{qu} with D_{ef} of form (8) in expression (25) for the diffusion velocity and to introduce, in the expression for the Lagrangian density, the additional term $U_T(q)\rho$ taking into account the energy density of the diffusion pressure because of the thermal environmental stochastic influence.

According to our reasoning, the expression for U_T must have a form that is similar to that of the factor $-mu^2/2$ in the cold vacuum (21). However, it must be modified so that $U_T \rightarrow 0$ as $T \rightarrow 0$. We introduce it as follows:

$$U_T(q) = -\frac{m}{2} \left[\frac{\alpha}{\Upsilon} \right]^2 u_{ef}^2 = -\frac{\hbar^2}{8m} \alpha^2 \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q} \right)^2, \quad (31)$$

$$\text{where } \alpha^2 \equiv \sinh^{-2} \kappa \frac{\omega}{T}; \quad \Upsilon \equiv \coth(\kappa \frac{\omega}{T}),$$

$$\text{and also } u_{ef} \equiv -D_{ef} \frac{1}{\rho} \frac{\partial \rho}{\partial q}$$

is the effective diffusion velocity in the thermal vacuum with D_{ef} of form (9), which is determined in analogy with the velocity u in (25).

Thus, as the Lagrangian density for $T \neq 0$, we chose the expression

$$\tilde{\mathcal{L}}_T(\rho, \theta) = -\hbar \frac{\partial \theta}{\partial t} \rho - \frac{m}{2}(v + u_{ef})^2 \rho - U\rho - U_T \rho. \quad (32)$$

For the convenience of further variation, we rewrite expression (32) in the explicit form in terms of the random functions θ and ρ :

$$\begin{aligned} \tilde{\mathcal{L}}_T(\rho, \theta) = -\hbar \frac{\partial \theta}{\partial t} \rho - & \left\{ \frac{\hbar^2}{2m} \left(\frac{\partial \theta}{\partial q} \right)^2 \rho - \frac{\hbar^2}{2m} \Upsilon \frac{\partial \theta}{\partial q} \frac{\partial \rho}{\partial q} + \frac{\hbar^2}{8m} \Upsilon^2 \frac{1}{\rho} \left(\frac{\partial \rho}{\partial q} \right)^2 \right\} - \\ & - U\rho - \frac{\hbar^2}{8m} \alpha^2 \frac{1}{\rho} \left(\frac{\partial \rho}{\partial q} \right)^2. \end{aligned} \quad (33)$$

Varying the action \mathcal{S} c $\tilde{\mathcal{L}}_T$ of form (33) with respect to θ again leads to the Fokker–Planck equation similar to (29) but with the replacement of D_{qu} with the effective diffusion coefficient D_{ef} in it:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} \left(\rho \frac{\hbar}{m} \frac{\partial \theta}{\partial q} \right) - D_{ef} \frac{\partial^2 \rho}{\partial q^2} = 0. \quad (34)$$

Accordingly, varying \mathcal{S} with respect to ρ leads to the Hamilton–Jacobi equation generalized to the case of the stochastic influence of the thermal vacuum:

$$\begin{aligned} \hbar \frac{\partial \theta}{\partial t} + \frac{\hbar^2}{2m} \left(\frac{\partial \theta}{\partial q} \right)^2 + \frac{\hbar^2}{2m} \Upsilon \frac{\partial^2 \theta}{\partial q^2} + U(q) - \\ - \frac{\hbar^2}{8m} \Xi_T \left[\frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial q} \right)^2 + 2 \frac{\partial}{\partial q} \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q} \right) \right] = 0, \end{aligned} \quad (35)$$

where

$$\Xi_T = 2\Upsilon^2 - 1 = 2 \coth^2(\varkappa \frac{\omega}{T}) - 1; \quad \Xi_0 = 1. \quad (36)$$

Obtained system of equations (34) and (35) generalizes Eqs. (22) and (23), allowing one to take the stochastic influence of the thermal vacuum into account consistently. It is indirectly represented in the quantities D_{ef} , Ξ_T , and Υ contained in these equations and dependent on the world constants \hbar and k_B .

Of course, the set of Fokker–Planck equations (34) and Hamilton–Jacobi equations (35) is a nontrivial generalization of the Schrödinger equations. They can be used in two ways. This system can be solved directly for the unknown functions ρ and θ . We showed recently [15] that this allows obtaining nonequilibrium wave functions whose amplitudes and phases are temperature-dependent and calculating macroparameters in the nonequilibrium states using them.

However, these equations can also be modified by representing them in the form of equations of two-velocity stochastic hydrodynamics for the characteristic velocities v and u . We see below that these equations are a generalization of the corresponding equations of Nelson’s stochastic mechanics [10].

6. One-dimensional model of two-velocity stochastic hydrodynamics

System of equations (32) and (33) can be solved directly as coupled equations for the dissimilar variables ρ and θ . However, the next step can be done, and these equations can be represented in the form of the equations for the similar variables, namely, the velocities v and u_{ef} , which are typical of any Markov processes. In this case, we obtain the system of equations of two-velocity stochastic hydrodynamics generalizing the equations of Nelson’s stochastic mechanics to the case of the quantum thermal environmental influence.

We now show that Eqs. (34) and (35) indeed allow obtaining the equations of stochastic hydrodynamics in the most convenient form. Taking the foregoing into account, we are now dealing only with one-dimensional model. To perform the corresponding transformation, we first represent continuity equation (34) in the form of an equation for the diffusion velocity, which can be written in the form

$$u_{ef} = -D_{ef} \frac{\partial \log \rho}{\partial q}$$

according to (25).

For this purpose, we first transform Eq. (34) by introducing v and u_{ef} explicitly in it and then multiply it by $(-D_{ef}/\rho)$:

$$-\frac{D_{ef}}{\rho} \cdot \frac{\partial \rho}{\partial t} - \frac{D_{ef}}{\rho} \left[\rho \frac{\partial(v + u_{ef})}{\partial q} + \frac{\partial \rho}{\partial q} (v + u_{ef}) \right] = 0. \quad (37)$$

Then we differentiate the result with respect to q , change the order of differentiation in the first term, and transform $\log \rho$, where it is possible, which allows introducing u_{ef} everywhere. As a result, we obtain

$$\frac{\partial u_{ef}}{\partial t} + \frac{\partial}{\partial q}(v u_{ef}) + \frac{\partial}{\partial q} u_{ef}^2 - D_{ef} \frac{\partial^2}{\partial q^2}(v + u_{ef}) = 0. \quad (38)$$

We can represent this equation in a more elegant form by forming the substantial derivative of the diffusion velocity u_{ef} , which is typical of hydrodynamics:

$$\frac{du_{ef}}{dt} \equiv \frac{\partial u_{ef}}{\partial t} + u_{ef} \frac{\partial u_{ef}}{\partial q} = -\frac{\partial}{\partial q}(v u_{ef}) - \frac{\partial}{\partial q} \frac{u_{ef}^2}{2} + D_{ef} \frac{\partial^2}{\partial q^2}(v + u_{ef}). \quad (39)$$

To represent Eq. (35) in the explicit hydrodynamic form, we also rewrite it in terms of the variables v and u_{ef} :

$$\hbar \frac{\partial \theta}{\partial t} + \frac{m}{2} v^2 + \frac{\hbar}{2} \Upsilon \frac{\partial v}{\partial q} + U(q) - \frac{m}{2} \Xi_T \cdot (u_{ef}^2 - \frac{\hbar}{m} \frac{\partial u_{ef}}{\partial q}) = 0. \quad (40)$$

To exclude the function θ , we differentiate Eq. (40) with respect to q , change the order of differentiation in the first term, and introduce the function v explicitly into it according to (24). As a result, also forming the substantial derivative of the drift velocity v in it, we obtain

$$\frac{dv}{dt} \equiv \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial q} \right) = -\frac{1}{m} \frac{\partial U}{\partial q} + \Xi_T \frac{\partial}{\partial q} \frac{u_{ef}^2}{2} - \frac{\hbar}{2m} \left(\Xi_T \frac{\partial^2 u_{ef}}{\partial q^2} + \Upsilon \frac{\partial v^2}{\partial q^2} \right). \quad (41)$$

We recall that, as the velocity v , only the quantity Δv generated by the stochastic action is contained in these equations. In the case under consideration, as at $T = 0$, the expressions for ρ and θ are related to wave functions of thermal correlated coherent states in which the exponent of the exponential depends on q^2 [5],[15]. It hence follows that the last terms in Eqs. (39) and (41) containing the second derivatives of $v \equiv \Delta v$ and u_{ef} with respect to q vanish.

As a result, the system of equations for the one-dimensional model of two-velocity stochastic hydrodynamics become

$$\begin{cases} \frac{du_{ef}}{dt} \equiv \frac{\partial u_{ef}}{\partial t} + \frac{\partial}{\partial q} \frac{u_{ef}^2}{2} = -\frac{\partial}{\partial q}(vu_{ef}) - \frac{\partial}{\partial q} \frac{u_{ef}^2}{2}; \\ \frac{dv}{dt} = -\frac{1}{m} \frac{\partial U}{\partial q} + \Xi_T \frac{\partial}{\partial q} \frac{u_{ef}^2}{2}. \end{cases} \quad (42)$$

Self-diffusion in the thermal vacuum characterized by the coefficient D_{ef} contained in u_{ef} is taken into account in both these equations. In addition, the right-hand side of the lower equation of the system contains not only the gradient of the classical potential $U(q)$, but also the gradient of the energy density of the diffusion pressure taking into account the stochastic influence of the quantum thermostat in the cases including the case $T = 0$. A similar contribution of the energy of the quantum-thermostat diffusion pressure is also contained in the right-hand side of the upper equation in (42); in this case, it does not vanish even at $v = 0$.

For the comparison with the equations of Nelson's stochastic mechanics

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial}{\partial q}(vu); \\ \frac{dv}{dt} = -\frac{1}{m} \frac{\partial U}{\partial q} + \frac{\partial}{\partial q} \frac{u^2}{2} \end{cases}, \quad (43)$$

we consider system of equations (42) in the case of the cold vacuum ($T = 0$) where u_{ef} transforms into u . In addition, for the convenience of comparison, we return to the partial time derivative in the upper equation of this system; to do this, we combine similar terms in formula (42). As a result, we have

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial}{\partial q}(vu) - \frac{\partial}{\partial q} \frac{u^2}{2}; \\ \frac{dv}{dt} = -\frac{1}{m} \frac{\partial U}{\partial q} + \frac{\partial}{\partial q} \frac{u^2}{2}. \end{cases} \quad (44)$$

As can be seen, the lower equations in systems (43) and (44) are identical completely. However, there is an important distinction between these systems of equations. As was expected, it is related to the fact that self-diffusion in the cold vacuum is taken into account in our theory; as a result of this, the equation for the diffusion velocity contains the gradient of the energy density of the diffusion pressure. Of course, unlike (43), our proposed equations (42) are applicable at any temperature in the general case.

7. Conclusions

Fenyés was, probably, the first who formulated the idea of using the Lagrangian density $\mathcal{L}[\rho; \theta]$ in the quantum theory [14]. The Fokker–Planck equation containing the total velocity V of the probability flux with the diffusion coefficient either D_{qu} or D_T follows from the his proposed expression. However, he did not introduce the generalized diffusion coefficient D_{ef} .

Unlike [14], in our approach, we consistently take into account quantum thermal fluctuations and the energy density of the diffusion pressure related to the stochastic environmental action (quantum thermostat) for $T \geq 0$. As a final result, we represent the Fokker–Planck and Hamilton–Jacobi equations in the form of system of equations (42) for the one-dimensional model of two-velocity stochastic hydrodynamics. The self-diffusion coefficient in them is determined by the effective environmental action, which is dependent on the fundamental constant \varkappa .

In our opinion, following this way, one can construct full-scale stochastic hydrodynamics taking into account not only self-diffusion but also shear viscosity and then use it to describe NPF. To do this, it is necessary to pass from the lower equation in (42) for the drift velocity to the equation that is a generalization of the Navier–Stokes equation to the case taking self-diffusion into account.

It follows from our analysis that the self-diffusion coefficient $D_{ef} = \mathbb{J}/m$ is, probably, the most adequate characteristic of transport phenomena, which is important for the description of dissipative processes in NPF. At present, it is possible to determine it experimentally by studying diffusion of massive quarks in quark–gluon plasma obtained in the process of heavy atom collisions.

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